We can also apply the free cocompletion property to prove that $PSh(C)$ (and in part. sSet) is cartesian closed.

def 14 A category D with finite products is cartesian closed is for all $a \in D$, the functor

$$a \times - : D \longrightarrow D$$

has a right adjoint, which is then denoted $\underline{\mathrm{Hom}}(a,-)$ and called the internal Hom or exponentiation. $\square$

Prop 15 Let C be a small category.

$PSh(C)$ is cartesian closed with

$$\boxed{\underline{Hom}(F,G)(X) = PSh(C)(F \times y(X), G)}$$

Proof: Can be done by hand, but also as applicat° of Thm 9.b):

- $F \times -$ is colimit-preserving (because products / colimits are computed objectwise) $\rightsquigarrow$ it has a right adjoint, given by this formula. $\square$

* In sSet, we get

$$\boxed{\underline{\mathrm{Hom}}(X_\bullet, Y_\bullet)_n = \mathrm{sSet}(X_\bullet \times \Delta^n, Y_\bullet)}$$

* For any cartesian closed category D and $X, Y, Z \in D$, there is a canonical composition

$$\underline{\mathrm{Hom}}(X, Y) \times \underline{\mathrm{Hom}}(Y, Z) \to \underline{\mathrm{Hom}}(X, Z).$$

<u>Exercise</u>: write it explicitely in $PSh(C)$.

3) Structure of Δ and applications

- We now go into the structure of Δ and what it means for sSet.

Notation:

Following [Rezk], we write

$$\delta = \langle \delta_0 \cdots \delta_n \rangle : [n] \longrightarrow [m]$$
$$k \longmapsto \delta_k$$

def 16 There are distinguished morphisms for every $0 \leq i \leq n$:

$$\delta^i := \langle 0 \cdots \hat{i} \cdots n \rangle : [n-1] \hookrightarrow [n]$$

(face morphisms)

$$\sigma^i := \langle 0 \cdots i\, i \cdots n \rangle : [n+1] \twoheadrightarrow [n]$$

(degeneracy morphisms)

If $X \in sSet$, we write

$$\begin{cases} d_i = (\delta^i)^* : X_n \longrightarrow X_{n-1} & \text{(face maps)} \\ s_i = (\sigma^i)^* : X_n \longrightarrow X_{n+1} & \text{(degeneracy maps)} \end{cases}$$

Lemma 17:

a) We have the simplicial identities:

$$\begin{cases} d_i d_j = d_{j-1} d_i , & i < j \\ s_i s_j = s_{j+1} s_i , & i \leq j \\ d_i s_j = \begin{cases} 1 , & i = j, j+1 \\ s_{j-1} d_i , & i < j \\ s_j d_{i-1} , & i > j+1 \end{cases} \end{cases}$$

b) Every morphism $\delta : [n] \to [m]$ can be written as

$$[n] \overset{S}{\twoheadrightarrow} [r] \overset{D}{\hookrightarrow} [m]$$

with $\begin{cases} S \text{ composite of deg. morphisms} \\ D \text{ ——— face morphisms} \end{cases}$

c) The datum of a simplicial object is equivalent to a diagram

$$\begin{tikzcd} X_0 \arrow[r, "s_0" description] & X_1 \arrow[l, bend right=40, "d_0"'] \arrow[l, bend left=40, "d_1"] \arrow[r, bend left=20, "s_0"] \arrow[r, bend right=20, "s_1"'] & X_2 \arrow[l, bend right=50, "d_0"'] \arrow[l, "d_1" description] \arrow[l, bend left=50, "d_2"] & \cdots \end{tikzcd} \qquad (*)$$

satisfying the simplicial identities.

□

<u>Proof</u>: a) is an exercise.

b) f factors uniquely into a surjective map follows by an injective map:

$$\begin{tikzcd}
{[n]} \arrow[r, two heads] \arrow[dr, two heads, "S"'] \arrow[r, phantom, shift right=4ex, "\equiv"] & \mathrm{Im}(f) \arrow[r, hook] \arrow[d, phantom, "\exists!\,\wr"] \arrow[r, phantom, shift right=4ex, "\equiv"] & {[m]} \\
 & {[r]} \arrow[ur, hook, "D"'] &
\end{tikzcd}$$

so it is enough to show that

- S is composition of deg. morphisms
- D ———————— faces ———

Let's do the case of S (D is similar)

This is an induction on $n - r \geq 0$.

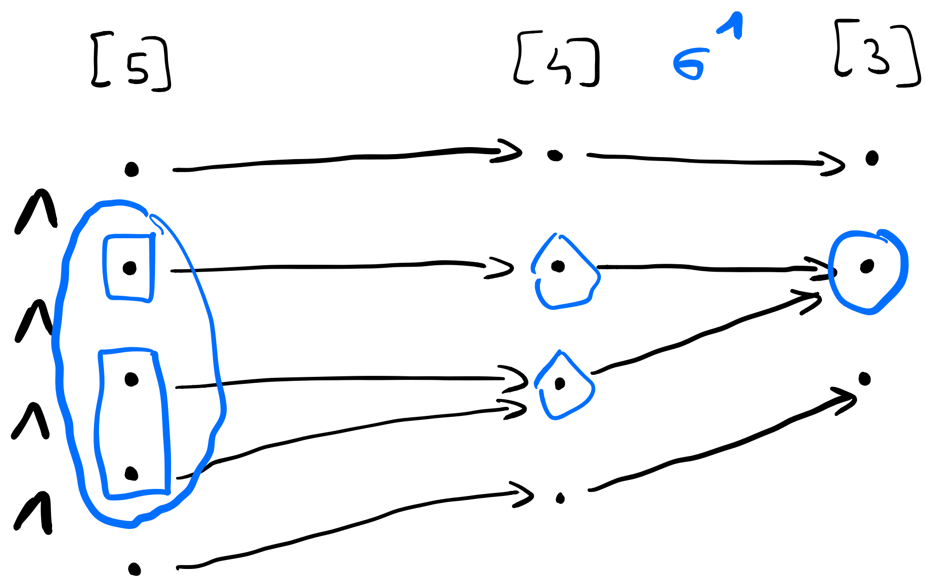
Assume $n > r$;

then $\exists\ i \in [r]$ such that $|D^{-1}(i)| > 1$.

Split $D^{-1}(i)$ into two non-empty sets to get a factorisation

$$D : [n] \xrightarrow{D'} [n+1] \xrightarrow{\sigma^i} [n] .$$

ex



c) Almost follows from a) + b) ; it remains to check that the simplicial identities imply that different choices of factorisations of

D, S yield the same map in a diagram $(*)$. $\square$

Rmk

a) Mnemotechnics

$$\underset{(i+1)}{\overset{d}{\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \vdots\\ \longleftarrow}}} \quad X_i \quad \underset{(i+1)}{\overset{D}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}}}$$

b) A cosimplicial object is similar:

$$X^0 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X^1 \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; X^2 \quad \cdots$$

$$\overset{1}{\longleftarrow} \qquad \overset{3}{\longrightarrow}$$

but imbalanced # of arrows...

def 18

- For $X_\bullet \in sSet$, a simplicial subset $Y_\bullet \subset X_\bullet$ is the datum of $\{Y_n \subset X_n\}_n$, stable under f^* for all $f : [m] \to [n]$ in Δ.

$\square$

Examples "Shape repertoire"!

* Let $S \subseteq [n]$. We write

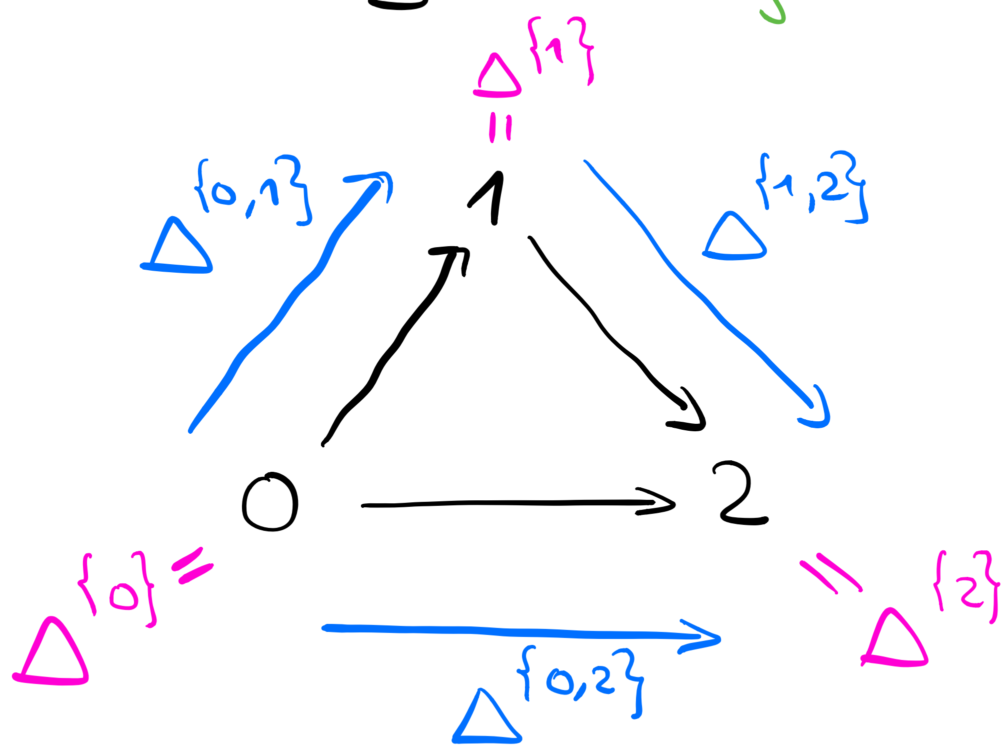
$\Delta^S \subseteq \Delta^n$ for the

S-face of Δ^n:

$$(\Delta^S)_k = \{ f \in (\Delta^n)_k \mid \operatorname{Im}(f) \subseteq S \}.$$

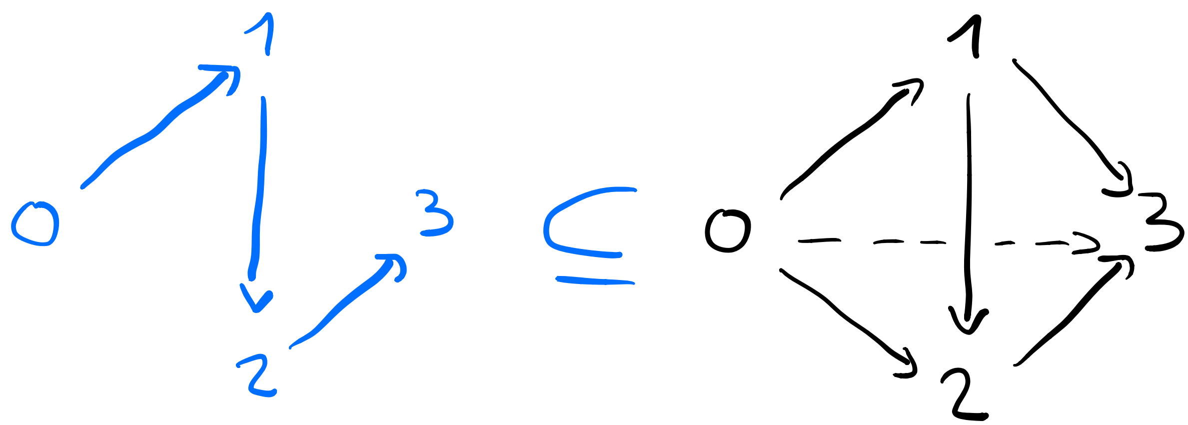
$$\exists ! \text{ isomorphism } \Delta^S \simeq \Delta^{|S|}.$$

$$\begin{cases} \Delta^{\{i\}} \subseteq \Delta^n \text{ are vertices} \\ \Delta^{\{i<j\}} \subseteq \Delta^n \text{ are edges} \end{cases}$$



* The spine $I^n \subseteq \Delta^n$ is

$$(I^n)_i = \{ \langle a_0 \cdots a_i \rangle \in (\Delta^n)_i \mid a_i \leq a_0 + 1 \}$$



$$I^3 \subseteq \Delta^3$$

* The boundary, or

simplicial n-sphere $\partial \Delta^n \subseteq \Delta^n$

is

$$(\partial\Delta^n)_i = \{ f \in (\Delta^n)_i \mid \mathrm{Im}(f) \neq [n] \}$$

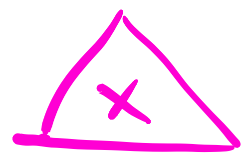
$\Rightarrow (\partial\Delta^n)_i = \Delta^n_i$

for $i < n$.

We have:

$$\partial\Delta^n = \bigcup_{j=0}^{n} \Delta^{[n]\setminus j}$$

$\partial\Delta^2$



The name comes from:

$$|\partial\Delta^n| \simeq S^{n-1}$$

* For $0 \leq k \leq n$, the k-th horn $\Lambda^n_k \subseteq \Delta^n$ is

$$(\Lambda^n_k)_i = \left\{ f \in (\Delta^n)_i \;\middle|\; \mathrm{Im}(f) \cup \{k\} \neq [n] \right\}$$

So:

$$\Lambda^n_k := \bigcup_{j \neq k} \Delta^{[n] \setminus j}$$

$$|\Lambda^n_k| \simeq D^{n-1} \quad (n-1)\text{-disk}$$

Λ^2_0	$\Lambda^2_1 \ (= I^2)$	Λ^2_2
$\begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$	$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}$	$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}$

* Horns Λ^n_k with

- $0 < k < n$ are inner horns.
- $k = 0, n$ are outer horns.

Lemma 20

a). Inner horns contain spines:

$$0 < k < n \implies I^n \subseteq \Lambda^n_k$$

b) $\forall n \geq 3$, $\begin{cases} I^n \subseteq \Lambda^n_0 \\ I^n \subseteq \Lambda^n_n \end{cases}$

c) · $I^1 \not\subseteq \Lambda^1_0, \Lambda^1_1$

$I^2 \not\subseteq \Lambda^2_0, \Lambda^2_2$ $\square$

4) Skeletal filtration

Presheaves are colimits of representables, but for simplicial sets we can be more precise.

Idea: Add first all the 0-simplices
then all the 1-simplices
...

def 21 Let $X_\bullet \in sSet$ and $x \in X_n$. We say that x is degenerate if $n > 0$ and the following equivalent conditions hold:

* $x \in \mathrm{Im}(s_i : X_{n-1} \to X_n)$ for some i.

* x factors through Δ^m for some $m < n$:

$$x : \Delta^n \longrightarrow \Delta^m \longrightarrow X_\bullet$$

Otherwise we say that x is non-degenerate. $\square$

Notation

- $X_n = X_n^{nd} \amalg X_n^{deg}$

⚠ $X_\bullet^{nd}, X_\bullet^{deg}$ are not simplicial subsets of $X_\bullet$.

· **Example**

$$(\Delta^n)_k^{nd} = \{ [k] \hookrightarrow [n] \}$$

correspond to the "true"

k-dimensional faces of $|\Delta^n|$.
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\wedge$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad|\Delta^s|$

Prop 22: (Eilenberg-Zilber lemma)

Let $X_\bullet \in sSet$, $n \geqslant 0$ and

$x \in X_n$. Then $x : \Delta^n \to X$

can be factored uniquely as

$$\boxed{x : \Delta^n \xrightarrow{y(p)} \Delta^m \xrightarrow{\tau} X_\bullet}$$

with p surjective, τ non-degenerate.

with :

- $p : [m] \twoheadrightarrow [n]$ surjective
- τ non-degenerate m-simplex .

proof :

Existence

Let $m \geq 0$ be minimal for the existence of a factorisation

$$x : \Delta^n \xrightarrow{y(g)} \Delta^m \xrightarrow{\tau} X .$$

Then :

- $y(g)$ is surjective (otherwise we would have a factorisation through $\mathrm{Im}(y(\delta))$ $) \iff \delta$ is surjective.

- τ is non-degenerate (otherwise we would have a factorisation

$$x : \Delta^n \to \Delta^m \to \Delta^{m'} \to X.$$

with $m' < m$).

<u>Uniqueness:</u>

Let $x : \Delta^n \xrightarrow{y(\delta')} \Delta^{m'} \xrightarrow{\tau'} X.$

be another such factorisation.

Write $\alpha = y(\delta)$, $\alpha' = y(\delta')$

* α, α' surjective

$$\Updownarrow$$

γ, γ' surjective

$$\Updownarrow$$

γ, γ' admit sections

$$\Updownarrow$$

α, α' admit sections β, β'.
$(\alpha \circ \beta = id, \ \alpha' \circ \beta' = id)$

* We get

$$\begin{aligned} \tau &= \tau \circ \alpha \circ \beta \\ &= x \circ \beta \\ &= \tau' \circ \alpha' \circ \beta . \end{aligned}$$

* Because τ is non-degenerate, $\alpha' \circ \beta$ inj $\Rightarrow m \leq m'$.

By symmetry, $m = m'$.

$\Rightarrow \alpha' \circ \beta = id_{[m]}$

$$\Rightarrow \begin{cases} \tau = \tau' \circ \alpha' \circ \beta = \tau' \\ \\ \alpha = \alpha' \circ \beta \circ \alpha = \alpha' \end{cases} \quad \blacksquare$$

def 23 (Skeleta)

$X_\bullet \in sSet$, $k \geq -1$.

$$Sk_k(X)_n := \left\{ x \in X_n, \exists \text{ fact}^\circ \; \Delta^n \to \Delta^m \to X_\bullet \text{ with } m \leq k \right\}$$

$Sk_k(X_\bullet) \subseteq X_\bullet$ is a simplicial subset of $X_\bullet$, the **k-th skeleton** of $X_\bullet$. $\square$

By construction:

$\coprod_{X_0} \Delta^0$ = discrete simplicial set $\hookrightarrow X_0$

$$\begin{cases} \emptyset = Sk_{-1}(X_\bullet) \subseteq Sk_0(X_\bullet) \subseteq Sk_1(X_\bullet) \subseteq \cdots \\[2em] \bigcup_{k \geq -1} Sk_k(X_\bullet) = X_\bullet \end{cases}$$

and $X_n^{nd} \cap Sk_k(X_\bullet) = \begin{cases} \emptyset, & k < n \\ X_n^{nd}, & k \geq n \end{cases}$

Rmk Sk_k induces a functor

$$Sk_k(-): sSet \longrightarrow sSet$$

with interesting properties (see Exercise Sheet 2).

Prop 24: Let $X_. \in sSet$, $k \geq 0$. There is a pushout square

$$\begin{tikzcd} \coprod_{X_k^{nd}} \partial\Delta^k \arrow[r, hook] \arrow[d] & \coprod_{X_k^{nd}} \Delta^k \arrow[d] \\ Sk_{k-1}(X_.) \arrow[r, hook] \arrow[ur, phantom, "\ulcorner"] & Sk_k(X_.) \end{tikzcd}$$

More generally, for any $A_\bullet \subseteq X_\bullet$ subcomplex, there is a pushout square

$$\begin{tikzcd} \coprod_{X_k^{nd} \setminus A_k^{nd}} \partial\Delta^k \arrow[r] \arrow[d] & \coprod_{X_k^{nd} \setminus A_k^{nd}} \Delta^k \arrow[d] \\ A_\bullet \cup Sk_{k-1}(X_\bullet) \arrow[r] & A_\bullet \cup Sk_k(X_\bullet) \end{tikzcd}$$

proof: Let's do the particular case $A_\bullet = \emptyset$.

We have $x \in X_k^{nd} \Rightarrow x \in Sk_k(X_\bullet)_k$.

and the faces of $x \in X_k^{nd}$ are in $Sk_{k-1}(X) \Rightarrow$ we have the commutative square of the statement.

- We observe that we have

$$\left(\coprod_{X_k^{nd}} \Delta^k\right)_n \setminus \left(\coprod_{X_k^{nd}} \partial\Delta^k\right)_n$$

$$\text{IS}$$

$$\left\{ \delta^* x \;\middle|\; x \in X_k^{nd},\ \delta : [n] \twoheadrightarrow [k] \right\}$$

IS (Eilenberg-Zilber)

$$Sk_k(X_\cdot)_n \setminus Sk_{k-1}(X_\cdot)_n .$$

- Moreover, the square is clearly a pullback.
- It remains to show

<u>Lemma</u> If
$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\ C \arrow[r, hook] & D \end{tikzcd}$$
in sSet

satisfies: $\forall n \geq 0$, $B_n \setminus A_n \xrightarrow{\sim} D_n \setminus C_n$

then it is a pushout.

which reduces to the same statement in Set (since (co)limits are computed objectwise) and is then an exercise. $\square$

Cor 25 Let $X_\bullet \in sSet$. Then the geometric realisation $|X_\bullet|$ is a CW-complex, whose k-cells are in bijection with X_k^{nd}.

If $A_\bullet \subseteq X_\bullet$ is a simplicial subset, then $|A_\bullet| \subseteq |X_\bullet|$ is a CW-subcomplex.

proof: $|-|$ is a left adjoint $\Rightarrow$ commutes with colimits.

Hence: $$|X_\bullet| = \bigcup_{k \geq -1} |Sk_k(X_\bullet)|$$

and we have a pushout

$$\begin{tikzcd}
\coprod_{X_k^{nd}} S^{k-1} \arrow[r] \arrow[d] & \coprod_{X_k^{nd}} D^k \arrow[d] \\
|Sk_{k-1}(X_\bullet)| \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & |Sk_k(X_\bullet)|
\end{tikzcd}$$

$\blacksquare$

Rmk This shows that, to model homotopy types, one could forget about degeneracies and work with $PSh(\Delta^{inj})$.

Not so for our purpose!

Examples

"I^n is a 1-dim simplicial set"

$$\begin{cases} \mathrm{Sk}_0(I^n) = \coprod \Delta^{\{i\}} \\ \mathrm{Sk}_i(I^n) = I^n \quad \text{for all } i \geq 1 \end{cases}$$

and $(I^n)_1^{nd} = \{ \Delta^{\{i,i+1\}} \mid 0 \leq i \leq n-1 \}$

We deduce :

$$\boxed{I^n = \Delta^{\{0,1\}} \coprod_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \coprod \cdots \Delta^{\{n-1,n\}}}$$

- Using Prop 24, can prove by induction on n :

$$\partial \Delta^n = \coprod_{\Delta^{[n]\setminus\{i,j\}}} \Delta^{[n]\setminus i}$$

and :

$$\Lambda^n_k = \coprod_{\substack{\Delta^{[n]\setminus\{i,j\}} \\ i \neq k}} \Delta^{[n]\setminus i}$$

$\Rightarrow$ explicit formulas for

$$sSet(I^n, X_\bullet) = \left\{ \begin{array}{l} a_0, \dots, a_n \in X_1 \\ d^1(a_i) = d^0(a_{i+1}) \end{array} \right\}$$

$$sSet(\partial\Delta^n, X_\bullet)$$

$$sSet(\Lambda^n_k, X_\bullet)$$

5) Kan complexes

Simplicial sets model homotopy types via geometric realisation. But if one tries to develop homotopy theory directly in sSet with $[0,1] \rightsquigarrow \Delta^1$ ("$|\Delta^1|$"), things do not work very well:

Rmk: In general, the relation on X_0 defined by

$$x \sim y \iff \exists\, \Delta^1 \xrightarrow{h} X, \quad d^0(h) = x,\ d^1(h) = y$$

is neither symmetric ($X = \Delta^1$)

nor transitive ($X = I^2$)

def 26 Let $X_\cdot \in sSet$.

$$\pi_0(X_\cdot) := X_0 / \simeq$$

set of connected components of $X_\cdot$

with $\simeq$ the equivalence relation generated by $\sim$. $\square$

Rmk $\pi_0(X_\cdot) \simeq \mathrm{Colim}(X_\cdot : \Delta^{op} \to Set)$

This is already unsatisfactory and things get worse for π_n for $n \geq 1$.

This is the topic of

simplicial homotopy theory

and leads to:

def 27 $X_\bullet \in sSet$ is a

Kan complex or Kan

simplicial set if it has

the Kan lifting property:

$\forall\, n \geq 1,\ \forall\ 0 \leq k \leq n,$

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \qquad \Bigg| \qquad \begin{tikzcd} X_n \arrow[d, two heads] \\ sSet(\Lambda^n_k, X_\bullet) \end{tikzcd}$$

$X_\bullet$ is a contractible Kan complex if $\forall n \geq 1$,

$$\begin{tikzcd} \partial \Delta^n \arrow[r] \arrow[d, hook] & X_\bullet \\ \Delta^n \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

More generally, $X_\bullet \to Y_\bullet$ is a Kan fibration if

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X_\bullet \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"'] & Y_\bullet \end{tikzcd}$$

and a trivial Kan fibration

if
$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & X_\bullet \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"'] & Y_\bullet \end{tikzcd}$$

Rmk It is not easy to really motivate these defs without going into simplicial homotopy theory.

The basic idea is:

"For a Kan complex $X_\bullet$, the

Homotopical properties of $|X_\cdot|$ can be expressed purely simplicially.

Recall:

$$|-| : sSet \rightleftarrows Top : Sing$$

with

$$Sing(A)_n = Top(\Delta^n_{top}, A)$$

Prop 28 Let $A \in Top$. Then the singular simplicial set $Sing(A)$ is a Kan complex.

Moreover,

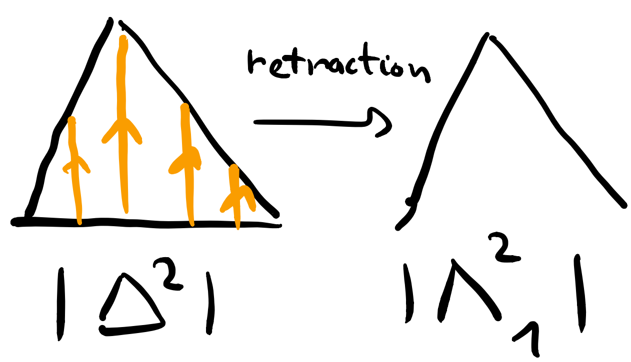
A is weakly contractible $\iff$ Sing(A) is a contractible Kan complex.

proof: By adjunction, we have to show:

$$\begin{tikzcd} {|\Lambda^n_k|} \arrow[r] \arrow[d, hook] & A \\ {|\Delta^n|} \arrow[ur, dashed, "\exists"'] & \end{tikzcd}$$

But $\overset{D^{n-1}}{\overset{\cong}{|\Lambda^n_k|}} \hookrightarrow \overset{D^n}{\overset{\cong}{|\Delta^n|}}$ admits a continuous retraction.

In pictures:



$$\begin{tikzcd}
{|\partial\Delta^n| \simeq S^{n-1}} \arrow[r] \arrow[d, hook] & A \\
{|\Delta^n| \simeq D^n} \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}$$
$\Leftrightarrow$ A weakly contractible. $\square$

def 29 The homotopy category of Kan complexes $h\mathrm{Kan}$

has : - objects = Kan complexes

- morphisms = Δ^1-homotopy classes (of morphisms in sSet

(need to show composition is well-defined)

- A morphism $f : X_\bullet \to Y_\bullet$ in sSet is a weak homotopy eq.

if $|f| : |X_\cdot| \to |Y_\cdot|$ is an homotopy equivalence of CW-complexes. $\square$

The main result of s. homotopy theory is:

Thm We have a diagram

$$\begin{tikzcd}
h\mathrm{Kan} \arrow[r, "|\cdot|", "\sim"'] \arrow[d, "s"] & h\mathrm{CW} \arrow[d, "s"] \\
s\mathrm{Set}[w.h.eq^{-1}] \arrow[r, "|\cdot|", "\sim"'] & \mathrm{Top}[w.h.eq^{-1}]
\end{tikzcd}$$

Remark This still does not explain why horns appear!

Lemma 30: Monomorphisms of simplicial sets are "generated" by the inclusions

$$\{\partial\Delta^n \hookrightarrow \Delta^n \mid n \in \mathbb{N}\}$$

under
- pushouts
- transfinite composition

proof Follows from the existence of the skeletal filtration and Proposition 24. $\square$

The fundamental reason that horns appear in the definition of Kan complexes is the analoguous result:

Prop 31: Monomorphisms of simpicial sets which are also weak homotopy equivalences are "generated by" the horn inclusions

$$\left\{ \Lambda^n_k \hookrightarrow \Delta^n \;\middle|\; \begin{array}{c} n \in \mathbb{N} \\ 0 \leq k \leq n \end{array} \right\}$$

under
- pushouts
- retracts
- transfinite composition.

$\square$